Chen *Fixed Point Theory and Applications* 2012, **2012**:41
http://www.fixedpointtheoryandapplications.com/content/2012/1/41

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Fixed point theorems for cyclic Meir-Keeler type mappings in complete metric spaces

Chi-Ming Chen

Correspondence: ming@mail.nhcue.edu.tw
Department of Applied Mathematics, National Hsinchu University of Education, No.521 Nanda Rd., Hsinchu City 300, Taiwan

Abstract

In this article, by using the Meir-Keeler type mappings, we obtain some new fixed point theorems for the cyclic orbital stronger (weaker) Meir-Keeler contractions and generalized cyclic stronger (weaker) Meir-Keeler contractions. Our results generalize or improve many recent fixed point theorems in the literature.

Mathematical Subject Classification: 54H25; 47H10

Keywords: generalized cyclic mapping, cyclic orbital mapping, fixed point theorem, cyclic Meir-Keeler contraction

1 Introduction and preliminaries

Throughout this article, by $\mathbb{R}^+$, we denote the set of all non-negative numbers, while $\mathbb{N}$ is the set of all natural numbers. It is well known and easy to prove that if (X, d) is a complete metric space, and if $f: X \to X$ is continuous and f satisfies

$$d(fx, f^2x) \leq k \cdot d(x, fx), \quad \text{for all } x \in X \text{ and } k \in (0, 1),$$

then f has a fixed point in X. Using the above conclusion, Kirk et al. [1] proved the following fixed point theorem.

Theorem 1 [1]*Let A and B be two nonempty closed subsets of a complete metric space (X, d), and suppose $f: A \cup B \to A \cup B$ satisfies*

(*i*) $f(A) \subset B$ *and* $f(B) \subset A$,
(*ii*) $d(fx, fy) \leq k \cdot d(x, y)$ *for all* $x \in A$, $y \in B$ *and* $k \in (0,1)$.

Then $A \cap B$ is nonempty and f has a unique fixed point in $A \cap B$.

The following definitions and results will be needed in the sequel. Let A and B be two nonempty subsets of a metric space (X, d). A mapping $f : A \cup B \to A \cup B$ is called a cyclic map if $f(A) \subseteq B$ and $f(B) \subseteq A$. In the recent, Karpagam and Agrawal [2] introduced the notion of cyclic orbital contraction, and obtained a unique fixed point theorem for such a map.

Definition 1 [2]*Let A and B be nonempty subsets of a metric space (X, d), $f : A \cup B \to A \cup B$ be a cyclic map such that for some $x \in A$, there exists a $\kappa_x \in (0,1)$ such that*

$$d(f^{2n}x, fy) \leq k_x \cdot d(f^{2n-1}x, y), \quad n \in \mathbb{N}, \quad y \in A. \tag{1}$$

Then f is called a cyclic orbital contraction.

Theorem 2 [2]*Let A and B be two nonempty closed subsets of a complete metric space* (X, d), *and let* $f : A \cup B \to A \cup B$ *be a cyclic orbital contraction. Then f has a fixed point in* $A \cap B$.

Furthermore, Kirk et al. [1] introduced the notion of the generalized cyclic mapping and obtained some fixed point results. Let $\{A_i\}_{i=1}^k$ be nonempty subsets of a metric space (X, d), and let $f : \cup_{i=1}^k A_i \to \cup_{i=1}^k A_i$ Then f is called a generalized cyclic map if $f(A_i) \subseteq A_{i+1}$ for $i = 1, 2,..., k$ and $A_{k+1} = A_1$. Kirk et al. [1] first extended the question of wherther Edelstein's [3] classical result for contractive mappings, and they obtained the following theorem.

Theorem 3 [1]*Let* $\{A_i\}_{i=1}^k$*be nonempty closed subsets of a complete metric space* (X, d), *at least one of which is compact, and suppose* $f : \cup_{i=1}^k A_i \to \cup_{i=1}^k A_i$*satisfies the following conditions (where* $A_{k+1} = A_1$*)*:

(*i*) $f(A_i) \subseteq A_{i+1}$ *for* $i = 1,2,...,k$,
(*ii*) $d(fx, fy) < d(x, y)$ *whenever* $x \in A_i$, $y \in A_{i+1}$ *and* $x \neq y$, $(i = 1, 2,..., k)$.

Then f has a unique fixed point.

On the other hand, Kirk et al. [1] took up the question of whether condition (*ii*) of Theorem 3 can be replaced by contractive conditions which typically arise in extensions of Banachs theorem. The authors began with a condition introduced by Geraghty [4]. Let S denote the class of those functions $\alpha : \mathbb{R}^+ \to [0,1)$ that satisfy the simple condition:

$$\alpha(t_n) \to 1 \quad \Rightarrow \quad t_n \to 0.$$

Theorem 4 [4]*Let* (X, d) *be a complete metric space, let* $f : X \to X$, *and suppose that there exists* $\alpha \lfloor\ S$ *such that*

$$d(fx, fy) \leq \alpha(d(x, y)) \cdot d(x, y), \quad \text{for all } x, y \in X.$$

Then f has a unique fixed point z in X and $\{f^n x\}$ *converges to z for each* $x \in X$.

Applying Theorem 4, Kirk et al. [1] proved the below theorem.

Theorem 5 [1]*Let* $\{A_i\}_{i=1}^k$*be nonempty closed subsets of a complete metric space (X, d), let* $\alpha \in \mathcal{S}$, *and suppose* $f : \cup_{i=1}^k A_i \to \cup_{i=1}^k A_i$*satisfies the following conditions* (*where* $A_{k+1} = A_1$):

(*i*) $f(A_i) \subseteq A_{i+1}$ *for* $i = 1,2,...,k$,
(*ii*) $d(fx, fy) \leq \alpha(d(x, y)) \cdot d(x, y)$ *for all* $x \in A_i$, $y \in A_{i+1}$, $i=1,2,...,k$.

Then f has a unique fixed point.

In 1969, Boyd and Wong [5] introduced the notion of Φ-contraction. A mapping $f : X \to X$ on a metric space is called Φ-contraction if there exists an upper semi-continuous function $\psi : [0,\infty) \to [0,\infty)$ such that

$$d(fx, fy) \leq \Phi(d(x, y)) \quad \text{for all } x, y \in X.$$

Kirk et al. [1] also proved the below theorem.

Theorem 6 [1]*Let* $\{A_i\}_{i=1}^k$*be nonempty closed subsets of a complete metric space* (X, d). *Suppose* $f : \cup_{i=1}^k A_i \to \cup_{i=1}^k A_i$*satisfies the following conditions (where* $A_{k+1} = A_1$*):*

(*i*) $f(A_i) \subseteq A_{i+1}$ *for* $i = 1,2,...,k$,
(*ii*) $d(fx, fy) \leq \Phi(d(x, y))$ *for all* $x \in A_i$, $y \in A_{i+1}$, $i = 1,2,...,k$,

where $\Phi : [0, \infty) \to [0, \infty)$ *is upper semi-contionuous from the right and satisfies* $0 \leq \psi(t) < t$ *for* $t > 0$. *Then f has a unique fixed point.*

In this article, we also recall the notion of the Meir-Keeler type mapping. A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a Meir-Keeler type mapping (see [6]), if for each $\eta \in \mathbb{R}^+$, there exists $\delta > 0$ such that for $t \in \mathbb{R}^+$ with $\eta \leq t < \eta + \delta$, we have $\psi(t) < \eta$. Subsequently, some authors worked on this notion (for example, [7-10]). This article will deal with two new mappings of the stronger Meir-Keeler type and weaker Meir-Keeler type in a metric space (X,d). We first introduce the below notion of stronger Meir-Keeler type mapping in a metric space.

Definition 2 *Let* (X, d) *be a metric space. We call* $\psi : \mathbb{R}^+ \to [0,1)$ *a stronger Meir-Keeler type mapping in X if the mapping* ψ *satisfies the following condition:*

$$\forall \eta > 0 \ \ \exists \delta > 0 \ \ \exists \gamma_\eta \in [0,1) \ \ \forall x, y \in X \ \ (\eta \leq d(x,y) < \delta + \eta \quad \Rightarrow \quad \psi(d(x,y)) < \gamma_\eta).$$

Example 1 *Let* $X = \mathbb{R}^2$ *and we define* $d : X \times X \to \mathbb{R}^+$ *by*

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2| \quad \text{for all } x = (x_1, x_2), \quad y = (y_1, y_2) \in X.$$

If $\psi : \mathbb{R}^+ \to [0,1)$, $\psi(d(x,y)) = \dfrac{d(x,y)}{d(x,y)+1}$, *then* ψ *is a stronger Meir-Keeler type mapping in X.*

The following provides an example of a Meir-Keeler type mapping which is not a stronger Meir-Keeler type mapping in a metric space (X, d).

Example 2 *Let* $X = \mathbb{R}^2$ *and we define* $d : X \times X \to \mathbb{R}^+$ *by*

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2| \quad \text{for all } x = (x_1, x_2), \quad y = (y_1, y_2) \in X.$$

If $\phi : \mathbb{R}^+ \to \mathbb{R}^+$,

$$\varphi(d(x,y)) = \begin{cases} d(x,y) - 1, & \text{if } d(x,y) > 1; \\ 0, & \text{if } d(x,y) \leq 1, \end{cases}$$

then ϕ *is a Meir-Keeler type mapping which is not a stronger Meir-Keeler type mapping in X.*

We next introduce the below notion of weaker Meir-Keeler type mapping in a metric space.

Definition 3 *Let* (X, d) *be a metric space, and* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$. *Then* ϕ *is called a weaker Meir-Keeler type mapping in X, if the mapping* ϕ *satisfies the following condition:*

$$\forall \eta > 0 \ \ \exists \delta > 0 \ \ \forall x, y \in X \ \ (\eta \leq d(x,y) < \delta + \eta \ \ \Rightarrow \ \ \exists n_0 \in \mathbb{N} \ \ \varphi^{n_0}(d(x,y)) < \eta).$$

Example 3 *Let* $X = \mathbb{R}^2$ *and we define* $d : X \times X \to \mathbb{R}^+$ *by*

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2| \quad \text{for all } x = (x_1, x_2), \quad y = (y_1, y_2) \in X.$$

If $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, $\varphi(d(x,y)) = \frac{1}{2}d(x,y)$, *then* ϕ *is a weaker Meir-Keeler type mapping in* X.

The following provides an example of a weaker Meir-Keeler type mapping which is not a Meir-Keeler type mapping in a metric space (X, d).

Example 4 *Let* $X = \mathbb{R}^2$ *and we define* $d : X \times X \to \mathbb{R}^+$ *by*

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2| \quad \text{for all } x = (x_1, x_2), \quad y = (y_1, y_2) \in X.$$

If $\phi : \mathbb{R}^+ \to \mathbb{R}^+$,

$$\varphi(d(x,y)) = \begin{cases} 0, & \text{if } d(x,y) \leq 1, \\ 2 \cdot d(x,y), & \text{if } 1 < d(x,y) < 2; \\ 1, & \text{if } d(x,y) \geq 2, \end{cases}$$

then ϕ *is a weaker Meir-Keeler type mapping which is not a Meir-Keeler type mapping in* X.

2 The fixed point theorems for cyclic orbital Meir-Keeler contractions

Using the notions of the cyclic orbital contraction (see, Definition 1) and stronger Meir-Keeler type mapping (see, Definition 2), we introduce the below notion of cyclic orbital stronger Meir-Keeler contraction.

Definition 4 *Let* A *and* B *be nonempty subsets of a metric space* (X, d). *Suppose* $f : A \cup B \to A \cup B$ *is a cyclic map such that for some* $x \in A$, *there exists a stronger Meir-Keeler type mapping* $\psi : \mathbb{R}^+ \to [0,1)$ *in* X *such that*

$$d(f^{2n}x, fy) \leq \psi(d(f^{2n-1}x, y)) \cdot d(f^{2n-1}x, y), \quad n \in \mathbb{N}, \quad y \in A. \tag{2}$$

Then f *is called a cyclic orbital stronger Meir-Keeler* ψ*-contraction.*

Now, we are in a position to state the following theorem.

Theorem 7 *Let* A *and* B *be two nonempty closed subsets of a complete metric space* (X, d), *and let* $\psi : \mathbb{R}^+ \to [0,1)$ *be a stronger Meir-Keeler type mapping in* X. *Suppose* $f : A \cup B \to A \cup B$ *is a cyclic orbital stronger Meir-Keeler* ψ*-contraction. Then* $A \cap B$ *is nonempty and* f *has a unique fixed point in* $A \cap B$.

Proof. Since $f : A \cup B \to A \cup B$ *is a* cyclic orbital stronger Meir-Keeler ψ-contraction, there exists $x \in A$ satisfying (2), and we also have that for each $n \in \mathbb{N}$,

$$\begin{aligned} d(f^{2n}x, f^{2n+1}x) &\leq \psi d(f^{2n-1}x, f^{2n}x)) \cdot d(f^{2n-1}x, f^{2n}x) \\ &\leq d(f^{2n-1}x, f^{2n}x), \end{aligned}$$

and

$$\begin{aligned} d(f^{2n+1}x, f^{2n+2}x) &= d(f^{2n+2}x, f^{2n+1}x) \\ &\leq \psi(d(f^{2n+1}x, f^{2n}x)) \cdot d(f^{2n+1}x, f^{2n}x) \\ &\leq d(f^{2n+1}x, f^{2n}x) = d(f^{2n}x, f^{2n+1}x). \end{aligned}$$

Generally, we have

$$d(f^n x, f^{n+1}x) \leq d(f^{n-1}x, f^n x), \quad n \in \mathbb{N}.$$

Thus the sequence $\{d(f^n x, f^{n+1}x)\}$ is non-increasing and hence it is convergent. Let $\lim_{n\to\infty} d(f^n x, f^{n+1}x) = \eta$. Then there exists $\kappa_0 \in \mathbb{N}$ and $\delta > 0$ such that for all $n \geq \kappa_0$,

$$\eta \le d(f^n x, f^{n+1}x) < \eta + \delta.$$

Taking into account the above inequality and the definition of stronger Meir-Keeler type mapping ψ *in X*, corresponding to η use, there exists $\gamma_\eta \in [0,1)$ such that

$$\psi(d(f^{k_0+n}x, f^{k_0+n+1}x)) < \gamma_\eta \quad \text{for all} \quad n \in \mathbb{N} \cup \{0\}.$$

Therefore, by (2), we also deduce that for each $n \in \mathbb{N}$,

$$\begin{aligned} d(f^{k_0+n}x, f^{k_0+n+1}x) &\le \psi(d(f^{k_0+n-1}x, f^{k_0+n}x)) \cdot d(f^{k_0+n-1}x, f^{k_0+n}x) \\ &< \gamma_\eta \cdot d(f^{k_0+n-1}x, f^{k_0+n}x), \end{aligned}$$

and it follows that for each $n \in \mathbb{N}$,

$$\begin{aligned} d(f^{k_0+n}x, f^{k_0+n+1}x) &< \gamma_\eta \cdot d(f^{k_0+n-1}x, f^{k_0+n}x) \\ &< \cdots\cdots \\ &< \gamma_\eta^n \cdot d(f^{k_0}x, f^{k_0+1}x). \end{aligned}$$

So

$$\lim_{n\to\infty} d(f^{k_0+n}x, f^{k_0+n+1}x) = 0, \quad \text{since} \quad \gamma_\eta \in [0,1).$$

We now claim that $\lim_{n\to\infty} d(f^{k_0+n}x, f^{k_0+m}x) = 0$ for $m > n$. For $m, n \in \mathbb{N}$ with $m > n$, we have

$$d(f^{k_0+n}x, f^{k_0+m}x) \le \sum_{i=n}^{m-1} d(f^{k_0+i}x, f^{k_0+i+1}x) < \frac{\gamma_\eta^{m-1}}{1-\gamma_\eta} d(f^{k_0}x, f^{k_0+1}x),$$

and hence $d(f^n x, f^m x) \to 0$, since $0 < \gamma_\eta < 1$. So $\{f^n x\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, there exists $\nu \in A \cup B$ such that $\lim_{n\to\infty} f^n x = \nu$. Now $\{f^{2n}x\}$ is a sequence in A and $\{f^{2n-1}x\}$ is a sequence in B, and also both converge to ν. Since A and B are closed, $\nu \in A \cap B$, and so $A \cap B$ is nonempty. Since

$$\begin{aligned} d(\nu, f\nu) &= \lim_{n\to\infty} d(f^{2n}x, f\nu) \\ &\le \lim_{n\to\infty} [\psi(d(f^{2n-1}x, \nu)) \cdot d(f^{2n-1}x, \nu)] \\ &\le \lim_{n\to\infty} [\gamma_\eta \cdot d(f^{2n-1}x, \nu)] = 0, \end{aligned}$$

hence ν is a fixed point of f.

Finally, to prove the uniqueness of the fixed point, let μ be another fixed point of f. By the cyclic character of f, we have $\nu, \mu \in A \cap B$. Since f is a cyclic orbital stronger Meir-Keeler ψ-contraction, we have

$$\begin{aligned} d(\nu, \mu) = d(\nu, f\mu) &= \lim_{n\to\infty} d(f^{2n}x, f\mu) \\ &\le \lim_{n\to\infty} [\psi(d(f^{2n-1}x, \mu)) \cdot d(f^{2n-1}x, \mu)] \\ &\le \lim_{n\to\infty} [\gamma_\eta \cdot d(f^{2n-1}x, \mu)] \\ &\le \gamma_\eta \cdot d(\nu, \mu) < d(\nu, \mu), \end{aligned}$$

a contradiction. Therefore $\mu = \nu$, and so ν is a unique fixed point of f.

Example 5 *Let* $A = B = X = \mathbb{R}^+$ *and we define* d: $X \times X \to \mathbb{R}^+$ *by*

$$d(x, y) = |x - y|, \quad \text{for } x, y \in X.$$

Define $f : X \to X$ *by*

$$f(x) = \begin{cases} 0, \text{ if } 0 \le x < 1; \\ \frac{1}{4}, \text{ if } x \ge 1. \end{cases}$$

and define $\psi : \mathbb{R}^+ \to [0,1)$ *by*

$$\psi(t) = \begin{cases} \frac{1}{3}, & \text{if } 0 \le t \le 1; \\ \frac{t}{t+1}, & \text{if } t > 1. \end{cases}$$

Then f is a cyclic orbital stronger Meir-Keeler ψ*-contraction and* 0 *is the unique fixed point.*

Using the notions of the cyclic orbital contraction (see, Definition 1) and weaker Meir-Keeler type mapping (see, Definition 3), we next introduce the notion of cyclic orbital weaker Meir-Keeler contraction. We first define the below notion of ϕ-mapping.

Definition 5 *Let* (X, d) *be a metric space. We call* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *a* ϕ*-mapping in X if the function* ϕ *satisfies the following conditions:*

(ϕ_1) ϕ *>is a weaker Meir-Keeler type mapping in X with* $\phi(0) = 0$;

(ϕ_2) (a) *if* $\lim_{n\to\infty} t_n = \gamma > 0$, *then* $\lim_{n\to\infty} \phi(t_n) < \gamma$, *and*

(b) *if* $\lim_{n\to\infty} t_n = 0$, *then* $\lim_{n\to\infty} \phi(t_n) = 0$;

(ϕ_3) $\{\phi^n(t)\}_{n\in\mathbb{N}}$ *is decreasing.*

Definition 6 *Let A and B be nonempty subsets of a metric space* (X, d). *Suppose* $f : A \cup B \to A \cup B$ *is a cyclic map such that for some* $x \in A$, *there exists a* ϕ*-mapping* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *in X such that*

$$d(f^{2n}x, fy) \le \varphi(d(f^{2n-1}x, y)), \quad n \in \mathbb{N}, \quad y \in A. \tag{3}$$

Then f is called a cyclic orbital weaker Meir-Keeler ϕ*-contraction.*

Now, we are in a position to state the following theorem.

Theorem 8 *Let A and B be two nonempty closed subsets of a complete metric space* (X, d), *and let* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a* ϕ*-mapping in X. Suppose* $f : A \cup B \to A \cup B$ *is a cyclic orbital weaker Meir-Keeler* ϕ*-contraction. Then* $A \cap B$ *is nonempty and f has a unique fixed point in* $A \cap B$.

Proof. Since $f : A \cup B \to A \cup B$ *is a* cyclic orbital weaker Meir-Keeler ϕ-contraction, there exists $x \in A$ satisfying (3), and we also have that for each $n \in \mathbb{N}$,

$$d(f^{2n}x, f^{2n+1}x) \le \varphi(d(f^{2n-1}x, f^{2n}x)),$$

and

$$\begin{aligned} d(f^{2n+1}x, f^{2n+2}x) &= d(f^{2n+2}x, f^{2n+1}x) \\ &\le \varphi(d(f^{2n+1}x, f^{2n}x)). \end{aligned}$$

Generally, we have

$$d(f^n x, f^{n+1}x) \le \varphi(d(f^{n-1}x, f^n x)), \quad n \in \mathbb{N}.$$

So we conclude that for each $n \in \mathbb{N}$

$$\begin{aligned} d(f^n x, f^{n+1}x) &\le \varphi(d(f^{n-1}x, f^n x)) \\ &\le \varphi^2(d(f^{n-2}x, f^{n-1}x)) \\ &\le \cdots\cdots \\ &\le \varphi^n(d(x, fx)). \end{aligned}$$

Since $\{\phi^n(d(x, fx))\}_{n\in\mathbb{N}}$ is decreasing, it must converge to some $\eta \ge 0$. We claim that $\eta = 0$. On the contrary, assume that $\eta > 0$. Then by the definition of weaker Meir-Keeler type mapping ϕ in X, there exists $\delta > 0$ such that for $x, y \in X$ with $\eta \le d(x, y) < \delta + \eta$, there exists $n_0 \in \mathbb{N}$ such that $\varphi^{n_0}(d(x, y)) < \eta$. Since $\lim_{n\to\infty} \phi^n(d(x, fx)) = \eta$, there exists $m_0 \in \mathbb{N}$ such that $\eta \le \phi^m(d(x, fx)) < \delta + \eta$, for all $m > m_0$. Thus, we conclude that $\varphi^{m_0+n_0}(d(x_0, x_1)) < \eta$, and we get a contradiction. So $\lim_{n\to\infty} \phi^n(d(x, fx)) = 0$, that is, $\lim_{n\to\infty} d(f^n x, f^{n+1}x) = 0$.

Next, we let $c_m = d(f^n x, f^{n+1}x)$, and we claim that the following result holds:

for each $\varepsilon > 0$, there is $n_0(\varepsilon) \in \mathbb{N}$ such that for all $m, n \ge n_0(\varepsilon)$,

$$d(f^m x, f^{m+1}x) < \varepsilon. \qquad (*)$$

We shall prove (*) by contradiction. Suppose that (*) is false. Then there exists some $\varepsilon > 0$ such that for all $p \in \mathbb{N}$, there are $m_p, n_p \in \mathbb{N}$ with $m_p > n_p \ge p$ satisfying:

(i) m_p is even and n_p is odd,
(ii) $d(f^{m_p}x, f^{n_p}x) \ge \varepsilon$, and
(iii) m_p is the smallest even number such that the conditions (*i*), (*ii*) hold.

Since $c_m \searrow 0$, by (*ii*), we have $\lim_{k\to\infty} d(f^{m_p}x, f^{n_p}x) = \varepsilon$, and

$$\begin{aligned} \varepsilon &\le d(f^{m_p}x, f^{n_p}x) \\ &\le d(f^{m_p}x, f^{m_p+1}x) + d(f^{m_p+1}x, f^{n_p+1}x) + d(f^{n_p+1}x, f^{n_p}x) \\ &\le d(f^{m_p}x, f^{m_p+1}x) + \varphi(d(f^{m_p}x, f^{n_p}x)) + d(f^{n_p+1}x, f^{n_p}x). \end{aligned}$$

Letting $p \to \infty$. Then by the condition (ϕ_2)-(a) of ϕ-mapping, we have

$$\varepsilon \le 0 + \lim_{p\to\infty} \varphi(d(f^{m_p}x, f^{n_p}x)) + 0 < \varepsilon,$$

a contradiction. So $\{f^n x\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, there exists $\nu \in A \cup B$ such that $\lim_{n\to\infty} f_n x = \nu$. Now $\{f^{2n}x\}$ is a sequence in A and $\{f^{2n-1} x\}$ is a sequence in B, and also both converge to ν. Since A and B are closed, $\nu \in A \cap B$, and so $A \cap B$ is nonempty. By the condition (ϕ_2)-(b) of ϕ-mapping, we have

$$\begin{aligned} d(\nu, f\nu) &= \lim_{n\to\infty} d(f^{2n}x, f\nu) \\ &\le \lim_{n\to\infty} \varphi(d(f^{2n-1}x, \nu)) = 0, \end{aligned}$$

hence ν is a fixed point of f. Let μ be another fixed point of f. Since f is a cyclic orbital weaker Meir-Keeler ϕ-contraction, we have

$$\begin{aligned} d(\nu,\mu) = d(\nu, f\mu) &= \lim_{n\to\infty} d(f^{2n}x, f\mu) \\ &\leq \lim_{n\to\infty} \varphi(d(f^{2n-1}x,\mu)) \\ &< d(\nu,\mu), \end{aligned}$$

a contradiction. Therefore, $\mu = \nu$. Thus ν is a unique fixed point of *f*.

Example 6 *Let* $A = B = X = \mathbb{R}^+$ *and we define* $d : X \times X \to \mathbb{R}^+$ *by*

$$d(x, y) = |x - y|, \quad \text{for } x, y \in X.$$

Define $f : X \to X$ *by*

$$f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1; \\ \frac{1}{4}, & \text{if } x \geq 1. \end{cases}$$

and define $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *by*

$$\varphi(t) = \frac{1}{3}t \quad \text{for } t \in \mathbb{R}^+.$$

Then f is a cyclic orbital weaker Meir-Keeler ϕ*-contraction and* 0 *is the unique fixed point.*

3 The fixed point theorems for generalized cyclic Meir-Keeler contractions

Using the notions of the generalized cyclic contraction [1] and stronger Meir-Keeler type mapping, we introduce the below notion of generalized cyclic stronger Meir-Keeler contraction.

Definition 7 *Let* $\{A_i\}_{i=1}^k$ *be nonempty subsets of a metric space* (X, d), *let* $\psi : \mathbb{R}^+ \to [0,1)$ *be a stronger Meir-Keeler type mapping in X, and suppose* $f : \cup_{i=1}^k A_i \to \cup_{i=1}^k A_i$ *satisfies the following conditions* (*where* $A_{k+1} = A_1$):

(i) $f(A_i) \subseteq A_{i+1}$ *for* $i = 1,2,\ldots,k$;
(ii) $d(fx, fy) \leq \psi(d(x, y)) \cdot d(x, y)$ *for all* $x \in A_i$, $y \in A_{i+1}$, $i=1,2,\ldots,k$.

Then we call f a generalized cyclic stronger Meir-Keeler ψ*-contraction.*

We state the main fixed point theorem for the generalized cyclic stronger Meir-Keeler ψ-contraction, as follows:

Theorem 9 *Let* $\{A_i\}_{i=1}^k$ *be nonempty closed subsets of a complete metric space* (X, d), *let* $\psi : \mathbb{R}^+ \to [0,1)$ *be a stronger Meir-Keeler type mapping in X, and let* $f : \cup_{i=1}^k A_i \to \cup_{i=1}^k A_i$ *be a generalized cyclic stronger Meir-Keeler* ψ*-contraction. Then f has a unique fixed point in* $\cap_{i=1}^k A_i$.

Proof. Given $x_0 \in X$ and let $x_n = f^n x_0$, $n \in \mathbb{N}$. Since *f* is a generalized cyclic stronger Meir-Keeler ψ-contraction, we have that for each $n \in \mathbb{N}$

$$\begin{aligned} d(x_n, x_{n+1}) &= d(f^n x_0, f^{n+1} x_0) \\ &\leq \psi(d(f^{n-1}x_0, f^n x_0)) \cdot d(f^{n-1}x_0, f^n x_0) \\ &\leq d(f^{n-1}x_0, f^n x_0) = d(x_{n-1}, x_n). \end{aligned}$$

Thus the sequence $\{d(x_n, x_{n+1})\}$ is non-increasing and hence it is convergent. Let $\lim_{n\to\infty} d(x_n, x_{n+1}) = \eta \geq 0$. Then there exists $\kappa_0 \in \mathbb{N}$ and $\delta > 0$ such that for all $n \geq \kappa_0$

$$\eta \leq d(x_n, x_{n+1}) < \eta + \delta.$$

Taking into account the above inequality and the definition of stronger Meir-Keeler type mapping ψ in X, corresponding to η use, there exists $\gamma_n \in [0,1)$ such that

$$\psi(d(x_{k_0+n}, x_{k_0+n+1})) < \gamma_\eta,$$

for all $n \in \mathbb{N} \cup \{0\}$. Thus, we can deduce that for each $n \in \mathbb{N}$

$$\begin{aligned} d(x_{k_0+n}, x_{k_0+n+1}) &= d(f^{k_0+n}x_0, f^{k_0+n+1}x_0) \\ &\leq \psi(d(f^{k_0+n-1}x_0, f^{k_0+n}x_0)) \cdot d(f^{k_0+n-1}x_0, f^{k_0+n}x_0) \\ &< \gamma_\eta d(f^{k_0+n-1}x_0, f^{k_0+n}x_0), \end{aligned}$$

and it follows that for each $n \in \mathbb{N}$

$$\begin{aligned} d(x_{k_0+n}, x_{k_0+n+1}) &< \gamma_\eta d(f^{k_0+n-1}x_0, f^{k_0+n}x_0) \\ &< \cdots \\ &< \gamma_\eta^n d(f^{k_0+1}x_0, f^{k_0+2}x_0). \end{aligned}$$

So

$$\lim_{n\to\infty} d(x_{k_0+n}, x_{k_0+n+1}) = 0, \quad \text{since} \quad \gamma_\eta < 1.$$

We now claim that $\lim_{n\to\infty} d(x_{k_0+n}, x_{k_0+m}) = 0$ for $m > n$. For $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} d(x_{k_0+n}, x_{k_0+m}) &= d(f^{k_0+n}x_0, f^{k_0+m}x_0) \\ &\leq \sum_{i=n}^{m-1} d(f^{k_0+i}x_0, f^{k_0+i+1}x_0) \\ &< \frac{\gamma_\eta^{m-1}}{1-\gamma_\eta} d(f^{k_0}x_0, f^{k_0+1}x_0)), \end{aligned}$$

and hence $d(f^n x_0, f^m x_0) \to 0$, since $0 < \gamma_\eta < 1$. So $\{f^n x_0\}$ is a Cauchy sequence. Since X is complete, there exists $\nu \in \cup_{i=1}^k A_i$ such that $\lim_{n\to\infty} f^n x_0 = \nu$. Now for all $i = 0,1, 2,..., k - 1$, $\{f^{kn-i}x\}$ is a sequence in A_i and also all converge to ν. Since A_i is clsoed for all $i = 1, 2,..., k$, we conclude $\nu \in \cap_{i=1}^k A_i$, and also we conclude that $\cap_{i=1}^k A_i \neq \phi$ Since

$$\begin{aligned} d(\nu, f\nu) &= \lim_{n\to\infty} d(f^{kn}x, f\nu) \\ &\leq \lim_{n\to\infty} [\psi(d(f^{kn-1}x, \nu)) \cdot d(f^{kn-1}x, \nu)] \\ &\leq \lim_{n\to\infty} [\gamma_\eta \cdot d(f^{kn-1}x, \nu)] = 0, \end{aligned}$$

hence ν is a fixed point of f.

Finally, to prove the uniqueness of the fixed point, let μ be another fixed point of f. By the cyclic character of f, we have $\mu \in \cap_{i=1}^k A_i$. Since f is a generalized cyclic stronger Meir-Keeler ψ-contraction, we have

$$\begin{aligned} d(\nu,\mu) &= d(\nu, f\mu) = \lim_{n\to\infty} d(f^{kn}x, f\mu) \\ &\le \lim_{n\to\infty}[\psi(d(f^{kn-1}x,\mu))\cdot d(f^{kn-1}x,\mu)] \\ &\le \lim_{n\to\infty}[\gamma_\eta \cdot d(f^{kn-1}x,\mu)] \\ &\le \gamma_\eta \cdot d(\nu,\mu) < d(\nu,\mu), \end{aligned}$$

a contradiction. Therefore, $\mu = \nu$. Thus ν is a unique fixed point of f.

Example 7 *Let* $X = \mathbb{R}^3$ *and we define* $d : X \times X \to \mathbb{R}^+$ *by*

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|, \quad \text{for } x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in X,$$

and let $A = \{(x, 0,0) : x \in \mathbb{R}\}$, $B = \{(0,y,0) : y \in \mathbb{R}\}$, $C = \{(0,0,z) : z \in \mathbb{R}\}$ *be three subsets of X. Define* $f: A \cup B \cup C \to A \cup B \cup C$ *by*

$$\begin{aligned} &f((x,0,0)) = (0,x,0); \quad \text{for all } x \in \mathbb{R}; \\ &f((0,y,0)) = (0,0,y); \quad \text{for all } y \in \mathbb{R}; \\ &f((0,0,z)) = (z,0,0); \quad \text{for all } z \in \mathbb{R}. \end{aligned}$$

and define $\psi : \mathbb{R}^+ \to [0,1)$ *by*

$$\psi(t) = \frac{t}{t+1} \quad ; \text{for } t \in \mathbb{R}^+.$$

Then f is a generalized cyclic stronger Meir-Keeler ψ*-contraction and* $(0,0,0)$ *is the unique fixed point.*

Using the notions of the generalized cyclic contraction and weaker Meir-Keeler type mapping, we introduce the below notion of generalized cyclic weaker Meir-Keeler contraction.

Definition 8 *Let* $\{A_i\}_{i=1}^k$ *be nonempty subsets of a metric space* (X, d), *let* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a* ϕ*-mapping in X, and suppose* $f : \cup_{i=1}^k A_i \to \cup_{i=1}^k A_i$ *satisfies the following conditions (where* $A_{k+1} = A_1$*):*

(*i*) $f(A_i) \subseteq A_{i+1}$ *for* $i = 1,2,...,k$;
(*ii*) $d(fx, fy) \le \phi(d(x, y))$ *for all* $x \in A_i$, $y \in A_{i+1}$, $i = 1,2,...,k$.

Then we call f a generalized cyclic weaker Meir-Keeler ϕ*-contraction.*

Now, we are in a position to state the following theorem.

Theorem 10 *Let* $\{A_i\}_{i=1}^k$ *be nonempty closed subsets of a complete metric space* (X, d), *let* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a* ϕ*-mapping in X, and let* $f : cup_{i=1}^k A_i \to \cup_{i=1}^k A_i$ *be a generalized cyclic weaker Meir-Keeler* ϕ*-contraction. Then f has a unique fixed point in* $\cap_{i=1}^k A_i$.

Proof. Given $x_0 \in X$ and let $x_n = f^n x_0$, $n \in \mathbb{N}$. Since f is a generalized cyclic weaker Meir-Keeler ϕ-contraction, we have that for each $n \in \mathbb{N}$

$$\begin{aligned} d(x_n, x_{n+1}) &= d(f^n x_0, f^{n+1} x_0) \\ &\le \varphi(d(f^{n-1}x_0, f^n x_0)) = \varphi(d(x_{n-1}, x_n)) \\ &\le \cdots\cdots \\ &\le \varphi^n(d(x_0, x_1)). \end{aligned}$$

Since $\{\phi^n(d(x_0,x_1))\}_{n\in\mathbb{N}}$ is decreasing, it must converge to some $\eta \geq 0$. We claim that $\eta = 0$. On the contrary, assume that $\eta > 0$. Then by the definition of weaker Meir-Keeler type mapping ϕ in X, there exists $\delta > 0$ such that for $x, y \in X$ with $\eta \leq d(x, y) < \delta + \eta$, there exists $n_0 \in \mathbb{N}$ such that $\varphi^{n_0}(d(x,y)) < \eta < \eta$. Since $\lim_{n\to\infty} \phi^n (d(x_0,x_1)) = \eta$, there exists $m_0 \in \mathbb{N}$ such that $\eta < \phi^m(d(x_0,x_1)) < \delta + \eta$, for all $m > m_0$. Thus, we conclude that $\varphi^{m_0+n_0}(d(x_0,x_1)) < \eta$, a contradiction. So $\lim_{n\to\infty} \phi^n(d(x_0,x_1)) = 0$, that is, $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Next, we claim that $\{x_n\}$ is a Cauchy sequence. We claim that the following result holds:

for each $\varepsilon > 0$, there is $n_0(\varepsilon) \in \mathbb{N}$ such that for all $m, n \geq n_0(\varepsilon)$,

$$d(x_m, x_n) < \varepsilon, \qquad (**)$$

We shall prove (**) by contradiction. Suppose that (**) is false. Then there exists some $\varepsilon > 0$ such that for all $p \in \mathbb{N}$, there are $m_p, n_p \in \mathbb{N}$ with $m_p > n_p \geq p$ satisfying:

(i) $d(x_{m_p}, x_{n_p}) \geq \varepsilon$, and
(ii) m_p is the smallest number greater than n_p such that the condition (i) holds.

Since

$$\begin{aligned}\varepsilon &\leq d(x_{m_p}, x_{n_p})\\ &\leq d(x_{m_p}, x_{m_{p-1}}) + d(x_{m_{p-1}}, x_{n_p})\\ &\leq d(x_{m_p}, x_{m_{p-1}}) + \varepsilon,\end{aligned}$$

hence we conclude $\lim_{p\to\infty} d(x_{m_p}, x_{n_p}) = \varepsilon$. Since

$$d(x_{m_p}, x_{n_p}) - d(x_{m_p}, x_{m_{p+1}}) \leq d(x_{m_p+1}, x_{n_p}) \leq d(x_{m_p}, x_{m_p+1}) + d(x_{m_p}, x_{n_p}),$$

we also conclude $\lim_{p\to\infty} d(x_{m_p+1}, x_{n_p}) = \varepsilon$. Thus, there exists i, $0 \leq i \leq k - 1$ such that m_p-n_p + i = 1 mod k for infinitely many p. If $i = 0$, then we have that for such p,

$$\begin{aligned}\varepsilon &\leq d(x_{m_p}, x_{n_p})\\ &\leq d(x_{m_p}, x_{m_p+1}) + d(x_{m_p+1}, x_{n_p+1}) + d(x_{n_p+1}, x_{n_p})\\ &\leq d(x_{m_p}, x_{m_p+1}) + \varphi(d(x_{m_p}, x_{n_p})) + d(x_{n_p+1}, x_{n_p}).\end{aligned}$$

Letting $p \to \infty$. Then by the condition (ϕ_2)-(a) of ϕ-mapping, we have

$$\varepsilon \leq 0 + \lim_{p\to\infty} \varphi(d(x_{m_p}, x_{n_p})) + 0 < \varepsilon,$$

a contradiction. The case $i \neq 0$ similar. Thus, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $\nu \in \cup_{i=1}^k A_i$ such that $\lim_{n\to\infty} x_n = \nu$. Now for all $i = 0, 1, 2, \ldots, k - 1$, $\{f^{kn-i}x\}$ is a sequence in A_i and also all converge to ν. Since A_i is closed for all $i = 1, 2, \ldots, k$, we conclude $\nu \in \cup_{i=1}^k A_i$, and also we conclude that $\cap_{i=1}^k A_i \neq \phi$. By the condition (ϕ_2)-(b) of ϕ-mapping, we have

$$\begin{aligned}d(\nu, f\nu) &= \lim_{n\to\infty} d(f^{kn}x, f\nu)\\ &\leq \lim_{n\to\infty} \varphi(d(f^{kn-1}x, \nu)) = 0,\end{aligned}$$

hence ν is a fixed point of f. Let μ be another fixed point of f. Since f is a generalized cyclic weaker Meir-Keeler ϕ-contraction, we have

$$\begin{aligned} d(\nu,\mu) &= d(\nu, f\mu) = \lim_{n\to\infty} d(f^{kn}x, f\mu) \\ &\le \lim_{n\to\infty} \varphi(d(f^{kn-1}x,\mu)) \\ &< d(\nu,\mu), \end{aligned}$$

a contradiction. Therefore, $\mu = \nu$. Thus ν is a unique fixed point of f.

Example 8 *Let* $X = \mathbb{R}^3$ *and we define* $d : X \times X \to \mathbb{R}^+$ *by*

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|, \text{ for } x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in X,$$

and let $A = \{(x,0,0): x \in \mathbb{R}\}$, $B = \{(0,y,0): y \in \mathbb{R}\}$, $C = \{(0,0,z): z \in \mathbb{R}\}$ *be three subsets of X. Define* $f : A \cup B \cup C \to A \cup B \cup C$ *by*

$$\begin{aligned} &f((x,0,0)) = \left(0, \frac{1}{4}x, 0\right); \quad \text{for all } x \in \mathbb{R}; \\ &f((0,y,0)) = (0,0,\frac{1}{4}y); \quad \text{for all } y \in \mathbb{R}; \\ &f((0,0,z)) = (\frac{1}{4}z, 0, 0); \quad \text{for all } z \in \mathbb{R}. \end{aligned}$$

and define $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *by*

$$\varphi(t) = \frac{1}{3}t \quad ; \text{for } t \in \mathbb{R}^+.$$

Then f is a generalized cyclic weaker Meir-Keeler ϕ*-contraction and* (0, 0, 0) *is the unique fixed point.*

Acknowledgements
The authors would like to thank referee(s) for many useful comments and suggestions for the improvement of the article.

Competing interests
The author declares that they have no competing interests.

Received: 3 December 2011 Accepted: 16 March 2012 Published: 16 March 2012

References

1. Kirk, WA, Srinivasan, PS, Veeramani, P: Fixed points for mappings satisfying cyclical contractive conditions. Fixed Point Theory. **4**(1):79–89 (2003)
2. Karpagam, S, Agrawal, S: Best proximity point theorems for cyclic orbital Meir-Keeler contraction maps. Nonlinear Anal. **74**, 1040–1046 (2010)
3. Edelstein, M: On fixed and periodic points under contractive mappings. J Lon Math Soc. **37**, 74–79 (1962). doi:10.1112/jlms/s1-37.1.74
4. Geraghty, MA: On contractive mappings. Proc Am Math Soc. **40**, 604–608 (1973). doi:10.1090/S0002-9939-1973-0334176-5
5. Boyd, DW, Wong, SW: On nonlinear contractions. Proc Am Math Soc. **20**, 458–464 (1969). doi:10.1090/S0002-9939-1969-0239559-9
6. Meir, A, Keeler, E: A theorem on contraction mappings. J Math Anal Appl. **28**, 326–329 (1969). doi:10.1016/0022-247X(69)90031-6
7. Di Bari, C, Suzuki, T, Vetro, C: Best proximity for cyclic Meir-Keeler contractions. Nonlinear Anal. **69**, 3790–3794 (2008). doi:10.1016/j.na.2007.10.014
8. Jankovic, S, Kadelburg, Z, Radonevic, S, Rhoades, BE: Best proximity point theorems for *p*-cyclic Meir-Keeler contractions. Fixed Point Theory Appl **2009**, 9 (2009). Article ID 197308

9. Suzuki, T: Fixed-point theorem for asymptotic contractions of Meir-Keeler type in complete metric spaces. Nonlinear Anal. **64**, 971–978 (2006). doi:10.1016/j.na.2005.04.054
10. Suzuki, T: Moudafis viscosity approximations with Meir-Keeler contractions. J Math Anal Appl. **325**, 342–352 (2007). doi:10.1016/j.jmaa.2006.01.080

doi:10.1186/1687-1812-2012-41
Cite this article as: Chen: Fixed point theorems for cyclic Meir-Keeler type mappings in complete metric spaces. *Fixed Point Theory and Applications* 2012 **2012**:41.